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Classification of Irreducible Integrable Modules for Multi-loop Algebras with Finite-Dimensional Weight Spaces

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INTRODUCTION

The purpose of this paper is to classify irreducible integrable modules for multi-loop algebras. Let V be a vector space over the complex numbers $\mathbb{C}$. Let $A = \mathbb{C}[t_1^{\pm}, \ldots, t_n^{\pm}]$ be the Laurent polynomials ring in n variables. Let $V_A = V \otimes A$ and let $v(\underline{m}) = v \otimes t^{\underline{m}}$ for $\underline{m} = (m_1, \ldots, m_n) \in \mathbb{Z}^n$ and $t^{\underline{m}} = t_1^{m_1} t_2^{m_2} \cdots t_n^{m_n}$.

Let $\mathcal{G}$ be a simple finite-dimensional Lie algebra over the complex numbers $\mathbb{C}$. Then $\mathcal{G}_A$ can be given a Lie algebra structure (see (1.1)). Let $d_1, \ldots, d_n$ be derivations defined by $[d_i, X(\underline{m})] = m_i X(\underline{m})$ and let D be the linear span of $d_1, \ldots, d_n$. Then $\widetilde{\mathcal{G}}_A = \mathcal{G}_A \oplus \mathcal{D}$ is a Lie algebra which we call a multi-loop algebra. The universal central extenion of $\mathcal{G}_A$ is called a toroidal Lie algebra and is studied in [BB, BC, E3, E4, EM]. So our multi-loop algebras are quotients of toroidal Lie algebras by centers.

Fix a Cartan subalgebra $\underline{h}$ of $\mathcal{G}$. Let $\bar{\psi}$ be a $\mathbb{Z}^n$-graded homomorphism of $U(\underline{h}_A) \to A$ (see Section 1 for details). Then we can define the universal highest weight module $M(\bar{\psi})$ for $\widetilde{\mathcal{G}}_A$. Let $V(\bar{\psi})$ be the unique irreducible quotient of $M(\bar{\psi})$. We then prove in Theorem 3.4 that any irreducible integrable module for $\widetilde{\mathcal{G}}_A$ is isomorphic to $V(\bar{\psi})$ by making use of a result from Chari [C].

We associate to $\bar{\psi}$ a map ψ from $U(h_A) \to \mathbb{C}$ by evaluating at 1 and an irreducible module $(V(\psi), \pi)$ for $\mathcal{G}_A$ (derivations do not act on $V(\psi)$, and therefore it cannot be extended to $\widetilde{\mathcal{G}}_A$). Now consider $V(\psi) \otimes A$ as a

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$\widetilde{\mathscr{G}}_A$-module by

$$X(\underline{m})v(\underline{r}) = \pi X(\underline{m})v(\underline{m}+\underline{r})$$
$$d_i v(\underline{r}) = r_i v(\underline{r})$$

for $X \in \mathscr{G}$, $v \in V(\psi)$, $\underline{m}, \underline{r} \in \mathbb{Z}^n$. We now observe that $V(\psi) \otimes A$ is a completely reducible $\widetilde{\mathscr{G}}_A$ module (Proposition 1.8) from a result in [E3]. We further prove in Proposition (1.9) that $V(\bar{\psi})$ is isomorphic to a component of $V(\psi) \otimes A$.

Assuming $V(\psi)$ is an integrable module, we prove in Proposition 3.6 that $V(\psi)$ is finite-dimensional. Then in Proposition 2.1 we prove that the finite-dimensional module is in fact a module for a semisimple Lie algebra. We conclude (Theorem 3.7) that any irreducible integrable module for $\widetilde{\mathscr{G}}_A$ comes in fact from a finite-dimensional module for a finite-dimensional semisimple Lie algebra (which is a quotient of $\mathscr{G}_A$). Such modules are easily classified and parameterized by dominant integral weights.

SECTION 1

Let $\mathscr{G}$ be simple finite-dimensional Lie algebra over $\mathbb{C}$. Then $\mathscr{G}_A$ can be made into a Lie algebra by defining

$$[X(\underline{m}), Y(\underline{s})] = [X, Y](\underline{m}+\underline{s}) \tag{1.1}$$

for $X, Y \in \mathscr{G}$, $\underline{m}, \underline{s} \in \mathbb{Z}^n$. Define derivations d_i, $1 \le i \le n$, by

$$[d_i, X(\underline{m})] = m_i X(\underline{m}).$$

Let D be the linear span of $d_1, \ldots, d_n$. Then $\widetilde{\mathscr{G}}_A = \mathscr{G}_A \oplus D$ is a Lie algebra which we call a multi-loop algebra. Fix a Cartan subalgebra $\underline{h}$ of $\mathscr{G}$. Let $\tilde{\underline{h}} = \underline{h} \oplus D$. Define null roots $\delta_i \in \tilde{\underline{h}}^*$ such that $\delta_i(\underline{h}) = 0$ and $\delta_i(d_j) = \delta_{ij}$. For $\underline{m} \in \mathbb{Z}^n$ let $\delta_{\underline{m}} = \sum m_i \delta_i$. For every positive root α of $\mathscr{G}$ let $X_\alpha, Y_\alpha, h_\alpha = [X_\alpha, Y_\alpha]$ be a $s\ell_2$-triple so that $[h_\alpha, X_\alpha] = 2X_\alpha$ and $[h_\alpha, Y_\alpha] = -2Y_\alpha$.

For any Lie algebra $\mathscr{G}'$, let $U(\mathscr{G}')$ be the universal enveloping algebra. Note that A, $\mathscr{G}_A$ and $U(\mathscr{G}_A)$ are $\mathbb{Z}^n$-graded.

Let $\bar{\psi}\colon U(\underline{h}_A) \to A$ be a $\mathbb{Z}^n$-graded Algebra homomorphism. Denote the image of $\bar{\psi}$ as $A_{\bar{\psi}}$. We make $A_{\bar{\psi}}$ into a $\tilde{\underline{h}}_A$-module by defining

$$h(\underline{r}) \cdot t^{\underline{m}} = \bar{\psi}(h(\underline{r})) \cdot t^{\underline{m}} \tag{1.2}$$
$$d_i \cdot t^{\underline{m}} = m_i t^{\underline{m}}$$

for $h \in \underline{h}$, $\underline{m}, \underline{r} \in \mathbb{Z}^n$.

(1.3) LEMMA (Lemma (1.2) of [E3]). *$A_{\bar{\psi}}$ is an irreducible $\underline{\tilde{h}}_A$-module if and only if every homogeneous element of $A_{\bar{\psi}}$ is invertible in $A_{\bar{\psi}}$.*

Let $\mathcal{G} = N^- \oplus \underline{h} \oplus N^+$, where N^+ is the sum of positive root spaces and N^- is the sum of negative root spaces. Let $\bar{\psi}$ as above. Let N_A^+ act trivially on $A_{\bar{\psi}}$. Now consider the following induced $\widetilde{\mathcal{G}}_A$-module. $M(\bar{\psi}) = U(\widetilde{\mathcal{G}}_A) \otimes_B A_{\bar{\psi}}$, where $B = N_A^+ \oplus \underline{\tilde{h}}_A$.

(1.4) PROPOSITION. (1) *As an $\tilde{h}$-module, $M(\bar{\psi})$ is a weight module.*

(2) *$M(\bar{\psi})$ is a free N_A^--module, and as a vector space*

$$M(\bar{\psi}) \cong U(N_A^-) \otimes_{\mathbb{C}} A_{\bar{\psi}}.$$

(3) *$M(\bar{\psi})$ has a unique irreducible quotient, which we call $V(\bar{\psi})$.*

Proof. (1) and (2) are standard. For (3), note that any proper submodule cannot intersect $A_{\bar{\psi}}$ (since $A_{\bar{\psi}}$ is irreducible). Now the sum of proper submodules is again a proper submodule, as it cannot intersect $A_{\bar{\psi}}$.

(1.5) DEFINITION. A $\widetilde{\mathcal{G}}_A$-module V is said to be the highest weight module if there exists a highest weight vector v such that

(1) $U(\widetilde{\mathcal{G}}_A)v = V$.

(2) $N_A^+ v = 0$.

(3) $U(\underline{\tilde{h}}_A)v \cong A_{\bar{\psi}}$ as an $\underline{\tilde{h}}_A$-module for some $\mathbb{Z}^n$-graded homomorphism $\bar{\psi}$.

Clearly any such highest weight module is a quotient of $M(\bar{\psi})$. We will also remark that if $\bar{\psi} = 0$ then $V(\bar{\psi})$ is the trivial module.

We also need the notion of a non-graded highest weight module.

(1.6) DEFINITION. A module W is said to be non-graded highest weight module if there exists a weight vector v in W such that

(1) $U(\mathcal{G}_A)v = W$.

(2) $N_A^+ v = 0$.

(3) There exists ψ in h_A^* such that $h.v = \psi(h)v$ for all h in h_A.

Let ψ belong to $\underline{h}_A^*$ and let $\mathbb{C}(\psi)$ be a one-dimensional vector space. Let N_A^+ act trivially on $\mathbb{C}(\psi)$ and let $\underline{h}_A$ act by ψ. Now consider the induced module for $\mathcal{G}_A$ given by $M(\psi) = U(\mathcal{G}_A) \otimes_{B^1} \mathbb{C}(\psi)$, where $B^1 = N_A^+ \oplus \underline{h}_A$. By standard argument one shows that $M(\psi)$ is an $\underline{h}$ weight module and has a unique irreducible quotient, which we will denote by $V(\psi)$.

Let $\bar{\psi}: U(\underline{h}_A) \to A_{\bar{\psi}}$ be a $\mathbb{Z}^n$-graded homomorphism. Consider $E_v(1): A_{\bar{\psi}} \to \mathbb{C}$ given by

$$E_v(1) \cdot t^{\underline{m}} = 1.$$

Let $(V(\psi), \pi)$ be the unique irreducible $\mathscr{G}_A$ module with respect to ψ. We will now make $V(\psi)_A$ into a $\widetilde{\mathscr{G}}_A$ module.

$$g(\underline{m}) \cdot v(\underline{r}) = \pi(g(\underline{m}))v(\underline{r} + \underline{m}), \tag{1.7}$$
$$d_i v(\underline{r}) = r_i v(\underline{r}).$$

It is now easy to check that (1.7) defines a module structure for $\widetilde{\mathscr{G}}_A$.

By using results from [E3] we will prove that $V(\bar{\psi})$ is isomorphic to a component of $V(\psi)_A$ as $\widetilde{\mathscr{G}}_A$-modules. Let $\bar{\psi}$ be as above and let $\psi = \bar{\psi} \circ E_v(1)$.

(1.8) PROPOSITION. *Let $A_{\bar{\psi}}$ be an irreducible $\tilde{h}_A$-module. Let $G \subseteq \mathbb{Z}^n$ be such that $\{t^{\underline{m}}, \underline{m} \in G\}$ is a set of coset representatives of $A/A_{\bar{\psi}}$. Let v be a highest weight vector of $V(\psi)$. Then*

(1) *$V(\psi)_A = \oplus_{\underline{m} \in G} Uv(\underline{m})$ as $\widetilde{\mathscr{G}}_A$-modules, where $Uv(\underline{m})$ is the submodule generated by $v(\underline{m})$.*

(2) *Each $U(v(\underline{m}))$ is an irreducible $\widetilde{\mathscr{G}}_A$-module.*

Proof. The proof follows from Theorem 1.8 of [E3]. There it is proved only for special $\bar{\psi}$, but the proof works for any $\bar{\psi}$ such that $A_{\bar{\psi}}$ is an irreducible $\tilde{h}_A$-module.

(1.9) PROPOSITION. *Assume $A_{\bar{\psi}}$ is an irreducible $\tilde{h}_A$-module. Then $V(\bar{\psi}) \cong U(v(0))$ as a $\widetilde{\mathscr{G}}_A$-module.*

Proof. Clearly $U(\underline{\tilde{h}}_A)v(0) \cong (A_{\bar{\psi}}, \bar{\psi})$ as a $\underline{\tilde{h}}_A$-module and $N_A^+ v(0) = 0$. Hence $U(v(0))$ is an irreducible (graded) highest weight module with highest weight $\bar{\psi}$. Hence by Proposition 1.4, $V(\bar{\psi}) \cong U(v(0))$.

We need the following lemma, which will be used in the next section.

(1.10) LEMMA. *Let $\psi, \bar{\psi}$ be as above. Then $V(\bar{\psi})$ has finite-dimensional weight spaces as an $\underline{\tilde{h}}$-module if and only if $V(\psi)$ has finite-dimensional weight spaces as an $\underline{h}$-module.*

Proof. Assume $V(\bar{\psi})$ has finite-dimensional weight spaces with respect to $\underline{\tilde{h}}$.

Let $\widetilde{G} = \{\underline{m} \mid t^{\underline{m}} \in A_{\bar{\psi}}\} \subseteq \mathbb{Z}^n$. Since $A_{\bar{\psi}}$ is an irreducible $\underline{\tilde{h}}_A$-module, by Lemma 1.3, $\widetilde{G}$ is a subgroup.

Case 1. $\widetilde{G}$ has a finite index in $\mathbb{Z}^n$. Suppose $V(\psi)_\lambda$ is infinite-dimensional for some λ. By the Poincaré–Birkhoff–Witt (PBW) theorem $V(\psi)_\lambda$ is spanned by vectors

$$Y_{\alpha_1}(\underline{m}_1) \cdots Y_{\alpha_k}(\underline{m}_k)v$$

(since $X_\alpha(\underline{m})$ kills v and $h(\underline{m})$ acts as a scalar), where $-\sum \alpha_i + \psi_0 = \lambda$ and $\psi \mid \underline{h} = \psi_0$. Suppose $Y_{\alpha_1}(\underline{m}_1) \cdots Y_{\alpha_k}(\underline{m}_k)v$ are linearly independent $(*)$. Since $\widetilde{G}$ is of finite index, we can assume that infinitely many $\sum \underline{m}_i$'s are in the same coset. Now choose $\underline{s}$ such that $-\sum m_i + \underline{s} \in \widetilde{G}$. Now choose $H(-\underline{m} + s) \in U(H)$ such that $\sum \underline{m}_i = \underline{m}$ and $\underline{\psi}(H(-\underline{m} + \underline{s})$ is nonzero and belongs to $A_{\underline{\psi}}$ of degree $-\underline{m} + \underline{s}$.

Consider

$$Y_{\alpha_1}(\underline{m}_1) \cdots Y_{\alpha_k}(\underline{m}_k) H(-\underline{m} + \underline{s})v, \qquad \sum \underline{m}_i = \underline{m},$$

which belongs to $V(\bar{\psi})_{\lambda + \delta_{\underline{s}}}$, a finite-dimensional space by assumption. Thus there exist scalars that are not all zero such that

$$\sum_{\substack{\alpha = (\alpha_1, \ldots, \alpha_k) \\ \bar{m} = (\underline{m}_1, \ldots, \underline{m}_k)}} a_{\alpha, \underline{m}} Y_{\alpha_1}(\underline{m}_1) \cdots Y_{\alpha_k}(\underline{m}_k) H(-\underline{m} + s)v = 0. \tag{1.11}$$

This clearly produces a nontrivial relation contradicting $(*)$.

Case 2. Index $\widetilde{G}$ is infinite in $\mathbb{Z}^n$. We can certainly assume that $h(\underline{n}) \neq 0$, and there exist X, Y root vectors such that $[X, Y] = h$, $[h, X] = 2X$, and $[h, Y] = -2Y$.

Choose infinitely many $\underline{m}$ such that they belong to distinct nontrivial cosets. In particular, $h(\underline{m})v = 0(**)$ for all $h \in \underline{h}$.

Claim 1. $Y(\underline{m})v$ is a linearly independent set in $V(\psi)$. Consider a relation

$$\sum a_{\underline{m}} Y(\underline{m})v = 0.$$

Then $X(\underline{r})$ has to kill it, which means

$$\sum a_{\underline{m}} h(\underline{m} + \underline{r}) = 0.$$

Now choose $\underline{r}$ such that $\underline{m}_1 + \underline{r} = \underline{n} \in \widetilde{G}$ for some $\underline{m}_1$. Then $\underline{m} + \underline{r}$ does not belong to $\widetilde{G}$ for all $\underline{m} \neq \underline{m}_1$. This implies $a_{\underline{m}_1} = 0$ by (**). This proves claim 1. It also proves that $Y(\underline{m})v$ is nonzero.

Claim 2. $Y(\underline{m})Y(-\underline{m} + \underline{r})v$ is a linearly independent set in $V(\bar{\psi})$ for some $\underline{r}$. Suppose a relation holds. Then there exist nonzero scalars $a_{\underline{m}, \underline{r}}$ such that

$$\sum a_{\underline{m}, \underline{r}} Y(\underline{m}) Y(-\underline{m} + \underline{r})v = 0.$$

Applying $X(\underline{s})$ to the above yields

$$\sum a_m Y(-\underline{m} + \underline{r}) h(\underline{m} + \underline{s}) - 2 \sum a_m Y(\underline{r} + \underline{s})$$
$$+ \sum a_m Y(\underline{m}) h(-\underline{m} + \underline{r} + \underline{s}) = 0.$$

Now choose $\underline{r} = 0, \underline{s} = \underline{n}$. Then as $Y(\underline{n})v$ is nonzero it follows (by **) that $\sum a_m = 0$. Now choose $\underline{s} \ni \underline{m}_1 + \underline{s} \in \widetilde{G}$ for some $\underline{m}_1$ and $\underline{m} + s$ does not belong to $\widetilde{G}$ for all $\underline{m} \neq \underline{m}_1$. Then by (**) we have $a_{\underline{m}_1} = 0$, a contradiction.

Thus in this case $V(\bar{\psi})$ cannot have finite-dimensional weight spaces.

Conversely assume that V is finite-dimensional for some λ in $V(\psi)$. Consider $V_\lambda \otimes A = \oplus_\delta (V_\lambda \otimes A)_{\lambda+\delta}$. Clearly each $(V_\lambda \otimes A)_{\lambda+\delta}$ is finite-dimensional. In particular $(Uv(0))_{\lambda+\delta} = U(v(0)) \cap (V_\lambda \otimes A)_{\lambda+\delta}$ is finite-dimensional.

We also need the following:

(1.12) LEMMA. *Let V be an irreducible $\mathcal{G}_A$-module with finite-dimensional weight spaces with respect to h. Then there exists a co-finite ideal I of A such that V is a module for $\mathcal{G} \otimes A/I$, a finite-dimensional Lie algebra.*

Proof. Let V_λ be a finite-dimensional weight space of V. Then V_λ is invariant under h_A. Since h_A is abelian, in particular solvable, by Lie's theorem there exists a nonzero v in V_λ such that

$$h(\underline{m})v = \lambda(h(m))v. \tag{1.13}$$

Fix a root α of $\mathcal{G}$ and consider the root vector X_α. Let i be fixed such that $1 \leq i \leq n$. Consider the set

$$\{X_\alpha(n_i)v, n_i \in \mathbb{Z}\} \subseteq V_{\lambda+\alpha}.$$

By assumption $V_{\lambda+\alpha}$ is finite-dimensional, so there exists a nonzero polynomial $P_i(\alpha) = \sum_j a_i t_i^j$ such that $X_\alpha \otimes P_i(\alpha)v = 0$ (here $X_\alpha \otimes P_i(\alpha) = \sum_j a_j X_\alpha \otimes t_i^j$). Let (P) denote the ideal generated by P inside of A.

Claim 1. $X_\alpha \otimes (t^{\underline{n}} P_i(\alpha))v = 0$. Consider the calculation

$$\begin{aligned} 0 &= h_\alpha(\underline{n}) X_\alpha \otimes P_i(\alpha)v = X_\alpha \otimes P_i(\alpha).h_\alpha(\underline{n})v + 2X_\alpha \otimes t^{\underline{n}} P_i(\alpha) \cdot v \\ &= \lambda(h_\alpha(n)) X_\alpha \otimes P_i(\alpha)v + 2X_\alpha \otimes t^{\underline{n}} P_i(\alpha) \cdot v. \end{aligned}$$

This proves the claim as $X_\alpha \otimes P_i(\alpha)v = 0$.

Claim 2. $h_\alpha \otimes (P_i(\alpha)$ and $P_i(-\alpha))v = 0$. Consider

$$\begin{aligned} 0 &= X_\alpha \otimes t^{\underline{n}} P_i(\alpha) \cdot X_{-\alpha} \otimes P_i(-\alpha)v - X_{-\alpha} \otimes P_i(-\alpha) X_\alpha \otimes t^{\underline{n}} P_i(\alpha)v \\ &= h_\alpha \otimes t^{\underline{n}} P_i(\alpha) P_i(-\alpha)v. \end{aligned}$$

This proves Claim 2.

Consider $\prod_\alpha P_i(\alpha) = P_i$. Then $\mathcal{G} \otimes (P_i)v = 0$. To see this just observe that $(P_i(\alpha)) \supseteq (P_i)$. Now let I be the ideal generated by $P_1, \ldots, P_n$. By the Chinese reminder theorem we see that I is co-finite in A. And the dimension of the quotient is bounded by the products of the degree P_i. Consider

$$W = \{v \in V \mid \mathcal{G} \otimes (P_I)v = 0\},$$

which is nonzero (by the above) and is a sub-module of V. Hence by irreducibility $W = V$. Thus the module is in fact a module for $\mathcal{G} \otimes A/I$.

SECTION 2

In this section we consider classification of finite-dimensional modules for $\mathcal{G}_A$. We will first construct finite-dimensional modules for $\mathcal{G}_A$. For $1 \leq i \leq k$ let $\underline{a}_i = (a_i(1), \ldots, a_i(n))$ be an n-tuple of nonzero complex numbers $\underline{m} \in \mathbb{Z}^n$ and let $a_i^{\underline{m}} = a_i(1)^{m_1} \cdots a_i(n)^{m_n}$ be the product. Consider the Lie algebra homomorphism $\varphi\colon \mathcal{G}_A \to \oplus \mathcal{G} = \mathcal{G}_k$($k$ copies) given by $\phi(X \otimes t^{\underline{a}}) = (a_i^{\underline{m}} X)$. φ is not surjective if and only if $a_i(\ell) = a_j(\ell)$ for some $i \neq j$ and for all ℓ (see [E3]). So when φ is surjective, any irreducible finite-dimensional module is an irreducible module for $\mathcal{G}_A$. Hence finite-dimensional modules exist. In this section we will prove that any finite-dimensional module has to come from the construction given above.

Let V be an irreducible finite-dimensional module for $\mathcal{G}_A$, with defining homomorphism

$$\pi\colon \mathcal{G}_A \to \cdot \text{ End V}.$$

Since V is finite-dimensional, Ker π is co-finite ideal in $\mathcal{G}_A$. It is easy to see that any ideal of $\mathcal{G}_A$ is of the form $\mathcal{G} \otimes I$ for some ideal I of A. So we have a map from $\mathcal{G} \otimes A/I \to$ End V, and $\mathcal{G} \otimes A/I$ is a finite-dimensional Lie algebra. We will now prove that the solvable radical of $\mathcal{G} \otimes A/I$ is zero on V.

(2.1) PROPOSITION. *Let V be an irreducible finite-dimensional module for a finite-dimensional Lie algebra $\mathcal{G} \otimes A/I$. Then the solvable radical R of $\mathcal{G} \otimes A/I$ is zero on V.*

Proof. Since R is an ideal, it is of the form $\mathcal{G} \otimes I'/I$ for some ideal I' of A. Since V is finite-dimensional and R is solvable, by Lie's theorem there exists a nonzero vector v in V such that

(2.2) $g \otimes P(t)v = \lambda(g, p(t))v$ for some scalar $\lambda(g, P(t))$, $g \in \mathcal{G}$, $P(t) \in I'$.

Let X_α be a root vector with α a root in $\mathcal{G}$.

Consider $(X_\alpha \otimes P(t))^m v = \lambda(X_\alpha, P(t))^m v$, since V is finite-dimensional $(X_\alpha \otimes P(t))^m v = 0$ for large m. This implies $\lambda(X_\alpha, P(t)) = 0$. Hence $X_\alpha \otimes P(t)v = 0$. Similarly $Y_\alpha \otimes P(t)v = 0$. We will now prove that

$$h_\alpha \otimes P(t)v = 0.$$

From (2.2) we have $h_\alpha \otimes P(t)v = \lambda v$ for $\lambda = \lambda(h_\alpha, P(t))$.

Claim 1. $h_\alpha \otimes P(t)Y_\alpha^m v = \lambda Y_\alpha^m v$. We prove this by induction on m. Let $m = 1$ and consider

$$h_\alpha \otimes P(t)Y_\alpha v = Y_\alpha h_\alpha \otimes P(t)v - 2Y_\alpha \otimes P(t)v = \lambda Y_\alpha v.$$

Assume the claim is true for m and consider

$$h_\alpha \otimes P(t)Y_\alpha^{m+1} v = Y_\alpha h_\alpha \otimes P(t)Y_\alpha^m v - 2Y_\alpha \otimes P(t)Y_\alpha^m v = \lambda Y_\alpha^{m+1} v$$

(by induction and the fact that $Y_\alpha \otimes P(t)v = 0$).

Claim 2. $X_\alpha \otimes P(t)Y_\alpha^m v = m\lambda Y_\alpha^{m-1} v$. The proof is by induction on m. Let $m = 1$ and consider the calculation

$$X_\alpha \otimes P(t)Y_\alpha v = Y_\alpha X_\alpha \otimes P(t)v + h_\alpha \otimes P(t)v = \lambda v.$$

Now assume the claim for m and consider the calculation

$$\begin{aligned} X_\alpha \otimes P(t)Y_\alpha^{m+1} v &= Y_\alpha X_\alpha \otimes P(t)Y_\alpha^m v + h_\alpha \otimes P(t)Y_\alpha^m v \\ &= m\lambda Y_\alpha^m v + \lambda Y_\alpha^m v = (m+1)\lambda Y_\alpha^m v. \end{aligned}$$

The claim is proved. Since V is finite-dimensional there exists $n_0 > 0$ such that $Y_\alpha^{n_0} v = 0$ and $Y_\alpha^{n_0 - 1} v = w \neq 0$. Consider

$$\begin{aligned} \lambda Y_\alpha^{n_0-1} v &= h_\alpha \otimes P(t)Y_\alpha^{n_0-1} v \qquad \text{by Claim 1} \\ &= [X_\alpha \otimes P(t), Y_\alpha]Y_\alpha^{n_0-1} v \\ &= (X_\alpha \otimes P(t)Y_\alpha - Y_\alpha X_\alpha \otimes P(t))Y_\alpha^{n_0-1} v \\ &= -\lambda(n_0 - 1)Y_\alpha^{n_0-1} v \end{aligned}$$

(by the choice of n_0 and Claim 2). This implies either $n_0 = 0$ or $\lambda = 0$. But by choice $n_0 \neq 0$ and hence $\lambda = 0$. Thus we have proved that $\mathcal{G} \otimes P(t)v = 0$ for $P(t) \in I'$. That is, $Rv = 0$. Now consider

$$W = \{v \in V; R.v = 0\}.$$

Clearly W is a nonzero sub-module of V. But V is irreducible; thus $W = V$ and $R.V = 0$. This completes the proof of the proposition.

We know that a finite-dimensional Lie algebra modulo its solvable radical is semi-simple. Thus a finite-dimensional irreducible module for $\mathcal{G}_A$ is actually a module for a semi-simple Lie algebra.

SECTION 3

Below we give classification of integrable modules for $\widetilde{\mathscr{G}}_A$ with finite-dimensional weight spaces.

(3.1) DEFINITION. *A $\widetilde{\mathscr{G}}_A$ module V is called integrable if*

(1) *V is a weight module with respect to $\tilde{h}$.*

(2) *For any nonzero root α, the elements $X_\alpha(\underline{n})$ act locally nilpotent; that is for every v in V there exists $N = N(\alpha, \underline{n}, v)$ such that $(X_\alpha(\underline{n}))^N v = 0$.*

(3.2) PROPOSITION. *Let V be an irreducible integrable module for $\widetilde{\mathscr{G}}_A$ with finite-dimensional weight spaces with respect to $\tilde{h}$. Then there exists a nonzero weight vector such that $N_A^+ v = 0$.*

Proof. The proof follows from Theorem 2.4(ii) of [C], where it was proved for $n = 1$, but the same proof will hold for any n. Just note that $(\lambda, \delta) = 0$ for any weight λ of V.

Let V be an irreducible integrable $\mathscr{G}_A$-module. Consider $\mathscr{G}_A = N_A^- \oplus \underline{h}_A \oplus N_A^+$. Then by the PBW theorem it will follow that for any highest weight vector v in V, $U(N_A^-)U(\underline{h}_A)v = V$. Let $W = U(\underline{h}_A)v$. Then W is an $\tilde{h}_A$-module. Since V is irreducible one can see that W is an irreducible $\tilde{h}_A$-module. Just observe that for any nonzero vector $U(\underline{h}_A)v$ is again a highest weight vector.

Let $I = \{X \in U(h_A) \mid X \cdot v = 0\}$. Then clearly $U(h_A)/I \cong W$ as an $\tilde{h}_A$-module. Since I is an ideal, and $U(h_A)$ is commutative, we conclude that W is a $\mathbb{Z}^n$-graded commutative simple algebra

(3.3) LEMMA. *Any graded simple commutative and associative algebra B is isomorphic to a subalgebra $A_{\bar{\psi}}$ of A for some $\bar{\psi}$ (as defined in Section 1). Furthermore, every nonzero homogeneous element in $A_{\bar{\psi}}$ is invertible in $A_{\bar{\psi}}$.*

Proof. Let X be a nonzero homogeneous element of B. Then XB is a $\mathbb{Z}^n$-graded ideal of B. Hence $XB = B$, so that X is invertible. Now consider B_0 as the zero component of B. Then any nonzero element $b \in B_0$ is invertible. Hence B_0 is a field. Furthermore, the dimension of each nonzero graded space is one as X defines an isomorphism between graded spaces. This proves the lemma. Thus $W \cong (A_{\bar{\psi}}, \bar{\psi})$ as a $\tilde{h}_A$-module. We have the following

(3.4) THEOREM. *Any integrable irreducible module for $\widetilde{\mathscr{G}}_A$ with finite-dimensional weight spaces is isomorphic to $V(\bar{\psi})$, where $\bar{\psi}$ is a $\mathbb{Z}^n$-graded homomorphism $U(h_A) \to A_{\bar{\psi}}$ and $A_{\bar{\psi}}$ is an irreducible $\tilde{h}_A$-module.*

Remark. For $n = 1$, this theorem is due to [C] and [CP]. See also [E1].

Recall $E_v(1)\colon V(\psi) \otimes A \to V(\psi)$ from Section 1.

(3.5) LEMMA. *Let W be a nonzero $\widetilde{\mathcal{G}}_A$-module of $V(\psi)_A$. Then $E_v(1)W = V(\psi)$.*

Proof. Since $V(\psi)$ is an irreducible $\mathcal{G}_A$-module, it is sufficient to prove that $E_v(1)W \neq 0$. But this is clear, as W contains a weight vector of the form $w(\underline{n})$ and $E_v(1) \cdot w(\underline{n}) = w$.

One can define integrable modules for $\mathcal{G}_A$. It is easy to see that if W is an integrable module for $\widetilde{\mathcal{G}}_A$ then $E_v(1)W$ is an integrable module for $\mathcal{G}_A$ (this is because $E_v(1)$ is a $\mathcal{G}_A$-module map).

(3.6) PROPOSITION. *Suppose $V(\bar{\psi})$ is an irreducible integrable module of $\widetilde{\mathcal{G}}_A$ with finite-dimensional weight spaces. Then $V(\psi)$ is finite-dimensional.*

Proof. We have already noted that $V(\psi)$ is integrable and irreducible for $\mathcal{G}_A$. By Lemma 1.10, $V(\psi)$ has finite-dimensional weight spaces. Now by Lemma 1.12 there exists a co-finite ideal I of A such that $V(\psi)$ is a module for $\mathcal{G} \otimes A/I$. By the PBW theorem we have $U(N^- \otimes A/I)U(h \otimes A/I)U(N^+ \otimes A/Iv) = V(\psi)$, where v is a highest weight vector of $V(\psi)$. Since v is a highest weight vector, $V(\psi) = U(N^- \otimes A/I)v$. But each vector of $N^- \otimes A/I$ acts locally nilpotently and hence $V(\psi)$ has to be finite-dimensional.

Remark. For $n = 1$ this proposition is due to [E1].

Thus we have proved the following:

(3.7) THEOREM. *Let V be an irreducible integrable module for $\widetilde{\mathcal{G}}_A$ with finite-dimensional weight spaces with respect to $\tilde{h}$. Then there exists $\bar{\psi}$: $U(h_A) \to A$, a $\mathbb{Z}^n$- graded homomorphism such that $V \cong V(\bar{\psi})$ as a $\tilde{\mathcal{G}}_A$- module. Furthermore, $V(\psi)$ is finite-dimensional. In fact $V(\psi)$ is an irreducible module for some finite-dimensional semisimple quotient of $\mathcal{G}_A$.*

Proof. The proof follows from Theorem 3.4, Proposition 3.6, and Proposition 2.1.

Note added in proof: In [E5] we have extended these results for toroidal Lie algebras, that is, for the universal central extension of $\mathcal{G}_A$. Furthermore, we have proved that the finite-dimensional semisimple Lie algebra that appears in Theorem 3.7 is exactly the one defined by φ in the first part of Section 2.

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